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Angular Momentum in Tensor Representations of $U(3)$

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Abstract

The problem of labelling the representations of $SO(3)$ contained in each representation of $U(3)$ is solved by using for this purpose an operator Z with integral eigenvalues. A description is given of the decomposition of any tensor representation of $U(3)$ into $SO(3)$–irreducible tensors labelled by these eigenvalues. An implicit definition of Z is given in terms of the $U(3)$ generators, and the relationship of Z to the labelling operators (with irrational eigenvalues) proposed by Racah and others is made clear. A possible application to a quark model of the hadrons is described.

1. *Introduction*

The intention of this paper is to present a solution of an important problem, of a group-theoretical nature, which arises in a variety of applications of quantum mechanics. Simply stated, the problem is to identify an additional quantum number, which in these applications is needed to distinguish the different tensor representations of the states of a system of particles, corresponding to given eigenvalues of the azimuthal and orbital angular momentum. It is well known that tensors constructed from the coordinate vectors or momenta of the system of particles can be resolved into irreducible representations of the unitary group $U(3)$. States with a given angular momentum, however, belong to irreducible representations of the orthogonal group $SO(3)$, and from the mathematical point of view the problem is therefore to find a suitable invariant Z to label the different irreducible representations of this group which occur within a given irreducible representation of $U(3)$. In the physical context, this invariant will correspond to an observable—which, for convenience, we here call the Z-spin—whose eigenvalue is one of the quantum numbers required to characterise a particular state.

It is important that the Z-spin always exists, but is not always a good

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quantum number. It will be so in applications with the symmetry of the unitary group $U(3)$, or its subgroup $SU(3)$. The first applications, and attempted solutions of the problems we have described, are found in articles written a decade or more ago, by Elliott (1958a, b), Bargmann & Moshinsky (1960, 1961) and Racah and his associates (Racah, 1962; Ilamed, 1968). Racah defined two scalar operators (denoted by x and y), either of which might serve as the 'missing' observable. But their eigenvalues proved to be irrational in general, and have never been determined except in special cases, though Hughes (1973a, b, c, d) has recently described a method by means of which he was able to compute a variety of numerical values. In a recent article, Louck & Galbraith (1972) have described a basis, for any given representation of $U(3)$, in which the orbital and azimuthal angular momentum operators are diagonal. However, there is no indication of what other operator is to be used to label the vectors in this basis, just as no observable has so far been found to correspond to the integral state label implicitly defined by Bargmann & Moshinsky (1960, 1961). Later in this paper, we shall give a relatively simple method of determining the eigenvalues of Racah's operators x and y, or the related operators O_l^0 and Q_l^0 defined by Hughes, and shall relate them to the integral parameter, thus implicitly defining a Z-spin with integral eigenvalues.

Interest in this problem has so far stemmed from special applications to atomic and low energy nuclear physics; since these are already discussed in the literature referred to above, we shall not describe them here. But we think it worth while to point out that the solution of the problem has important implications for non-relativistic quantum mechanics in general, and not just those applications involving harmonic oscillators, etc., where an $SU(3)$ symmetry is immediately apparent. In the next section we shall develop this theme for an arbitrary quantum mechanical system. There is, however, another kind of application which appears so far to have escaped attention, and which we shall outline in the third section. This concerns elementary particle physics, and especially the theory of the hadrons, whether based on a quark model or not. In Gell-Mann's well known scheme (1962), the isospin is associated with the $SU(2)$ subgroup of the unitary group $U(3)$, and the hypercharge Y is an observable whose eigenvalues distinguish the $SU(2)$ submultiplets in an irreducible representation of $U(3)$. But there is an alternative and interesting possibility, in which *two* isospin multiplets are associated with each irreducible representation of $SO(3)$ within any irreducible representation of $U(3)$. As the authors have pointed out elsewhere (Bracken & Green, 1973), this possibility is inherent in a recent suggestion by Govorkov (1968, 1969, 1973) that the unitary spin observables are invariants of a generalised parastatistics algebra. A quark model can be developed, in which the hypercharge is closely associated with the Z-spin, in this particular application.

In the final section, we present a solution of the algebraic problem, which is fundamental to the physical application described, of defining the Z-spin in terms of the generators of $U(3)$.

2. *The Unitary Group in Quantum Mechanics*

Let us consider a system of particles, whose states may be represented in terms of a set of three-dimensional vectors ξ_I ($I = 1, 2, \ldots$) with components ξ_I^i ($i = 1, 2, 3$) relative to rectilinear but not necessarily orthogonal axes. The vectors could be position vectors, or momenta, or other vectors with numerical components.

A very useful type of representation in terms of such vectors is furnished by tensors of the form

$$T^{ij..z} = \xi_I^i \xi_J^j \ldots \xi_Z^z S(\boldsymbol{\xi}) \tag{2.1}$$

where $S(\boldsymbol{\xi})$ is a scalar function of the $\boldsymbol{\xi}$, the same for all tensors of the type considered. A tensor such as $T^{ij..z}$ may be regarded as a representation of the unitary group $U(3)$, though not in general an irreducible representation. The generators of the group may be defined by

$$a_t^s = \Sigma_I \xi_I^s \left(\frac{\partial}{\partial \xi_I^t} - \frac{1}{S(\boldsymbol{\xi})} \frac{\partial S(\boldsymbol{\xi})}{\partial \xi_I^t} \right) \tag{2.2}$$

in this representation and they are obviously linear differential operators. It is convenient to define also three independent invariants λ_1, λ_2, and λ_3 of the group by (see Green, 1971)

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= a^s{}_s \\
(\lambda_1 + 1)^2 + \lambda_2^2 + (\lambda_3 - 1)^2 &= a^s{}_t a^t{}_s + 2 \\
(\lambda_1 + 1)^3 + \lambda_2^3 + (\lambda_3 - 1)^3 &= a^s{}_t a^t{}_u a^u{}_s + 2a^s{}_s - \tfrac{1}{2}(2a^s{}_t a^t{}_s - a^s{}_s a^t{}_t)
\end{aligned} \tag{2.3}$$

where summation over the repeated affixes s, t and u is understood; the eigenvalues l_1, l_2 and l_3 of λ_1, λ_2 and λ_3 in any irreducible representation are then integers and λ_1, λ_2 and λ_3 can be chosen so that $l_1 \geqslant l_2 \geqslant l_3$. The decomposition of the tensor $T^{ij..z}$ into irreducible representations of $U(3)$ can be effected conveniently by making use of the identity (from Green, 1971)

$$\begin{aligned}
a_1{}^i{}_j a_2{}^j{}_k a_3{}^k{}_l &= 0 \\
a_1{}^i{}_j &= a^i{}_j - (\lambda_1 + 2)\delta^i{}_j \\
a_2{}^i{}_j &= a^i{}_j - (\lambda_2 + 1)\delta^i{}_j \\
a_3{}^i{}_j &= a^i{}_j - \lambda_3 \delta^i{}_j
\end{aligned} \tag{2.4}$$

This allows us to resolve any vector $\boldsymbol{\xi}_I$ into three vector operators:

$$\boldsymbol{\xi}_I = \boldsymbol{\xi}_{1I} + \boldsymbol{\xi}_{2I} + \boldsymbol{\xi}_{3I} \tag{2.5}$$

such that

$$\lambda_r \boldsymbol{\xi}_{sI} = \boldsymbol{\xi}_{sI}(\lambda_r + \delta_{rs}) \tag{2.6}$$

Explicitly,

$$\xi_{1I}^{\ i} = f_1 a_{2\ j}^{\ i} a_{3\ k}^{\ j} \xi^k{}_I$$
$$f_1 = [(\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_3 + 2)]^{-1} \tag{2.7}$$

etc. Obviously any tensor of the type

$$T_c^{ij..z} = \xi_r{}^i{}_I \xi_s{}^j{}_J \ldots \xi_u{}^z{}_Z S(\boldsymbol{\xi})$$

if it does not vanish, will correspond to an irreducible representation of $U(3)$. We shall consider in particular the tensor

$$T_c^{ij..z} = \xi_1{}^i{}_I \xi_1{}^j{}_J \ldots (\xi_2{}^p{}_P \xi_1{}^q{}_Q)(\xi_2{}^r{}_R \xi_1{}^s{}_S) \ldots (\xi_3{}^x{}_X \xi_2{}^y{}_Y \xi_1{}^z{}_Z) S(\boldsymbol{\xi}) \tag{2.8}$$

in which there are $l_1 - l_2$ factors like $\xi_1{}^i{}_I$, $l_2 - l_3$ factors like $(\xi_2{}^p{}_P \xi_1{}^q{}_Q)$ and l_3 factors like $(\xi_3{}^x{}_X \xi_2{}^y{}_Y \xi_1{}^z{}_Z)$, in that order. Then $T_c^{ij..z}$ will be an eigenvector of λ_1, λ_2 and λ_3 corresponding to the eigenvalues l_1, l_2 and l_3, and is an irreducible representation of $U(3)$ labelled (l_1, l_2, l_3).

Now, it is well known that irreducible tensor representations of $U(3)$ have specific symmetries. The tensor $T_c^{ij..z}$ can be derived from $T^{ij..z}$ by first symmetrising with respect to superscripts in each of the sets $\{i, j, \ldots q, s, \ldots z\}$, $\{p, r, \ldots y\}$, $\{\ldots x\}$, associated with the subscripts 1, 2, 3 in (2.8), then antisymmetrising with respect to superscripts in each of the sets $\{p, q\}$, $\{r, s\}, \ldots \{x, y, z\}$ associated with different factors in (2.8). One result of this is that a factor like $(\xi_3{}^x{}_X \xi_2{}^y{}_Y \xi_1{}^z{}_Z)$ can be replaced by $\epsilon^{xyz}\sigma_{XYZ}$, where ϵ^{xyz} is the alternating tensor and

$$\sigma_{XYZ} = \epsilon_{xyz}(\xi_3{}^x{}_X \xi_2{}^y{}_Y \xi_1{}^z{}_Z)/6 \tag{2.9}$$

is a pseudoscalar. Also, a factor like $(\xi_2{}^p{}_P \xi_1{}^q{}_Q)$ can be replaced by $\epsilon^{pqu}\eta_{PQu}$, where

$$\eta_{PQu} = \epsilon_{pqu}(\xi_2{}^p{}_P \xi_1{}^q{}_Q)/2 \tag{2.10}$$

Thus, $T_i^{ij..z}$ is reduced to the form

$$T_c^{ij..z} = \epsilon^{pqu}\epsilon^{rsv} \ldots \epsilon^{xyz} U_c^{ij..}{}_{uv..}$$
$$U_c^{ij..}{}_{uv..} = \xi_1{}^i{}_I \xi_1{}^j{}_J \ldots \eta_{PQu}\eta_{RSv} \ldots S' \tag{2.11}$$

where S' is a scalar or pseudoscalar resulting from the absorption of the factors σ_{XYZ} in S. We shall therefore consider tensors of the type $U_c^{ij..}{}_{uv..}$, which belong to the same representation (l_1, l_2, l_3) of $U(3)$ as $T_c^{ij..z}$. It is important to notice that $U_c^{ij..}{}_{uv..}$ is totally symmetric in both superscripts and subscripts, and satisfies

$$\delta^u{}_i U_c^{ij..}{}_{uv..} = 0 \tag{2.12}$$

The problem with which we are here concerned is the decomposition of $T_c^{ij..z}$, or equivalently $U_c^{ij..}{}_{uv..}$, into independent angular momentum states, which correspond to irreducible representations of $SO(3)$. The orbital angular momentum L is related to the generators of $U(3)$ by

$$L(L+1) = l^i{}_j l^j{}_i$$
$$l^i{}_j = a^i{}_j - g^{il} a^k{}_l g_{kj} \tag{2.13}$$

where g_{ij} is the metric tensor and $g^{ij} g_{jk} = \delta^i{}_k$. It is known that the tensor operator $l^i{}_j$ satisfies the identity (Bracken & Green, 1971)

$$l_0{}^i{}_j l_+{}^i{}_j l_-{}^i{}_j = 0 \tag{2.14}$$

where

$$l_0{}^i{}_j = l^i{}_j - \delta^i{}_j$$
$$l_+{}^i{}_j = l^i{}_j - (L+1)\delta^i{}_j$$
$$l_-{}^i{}_j = l^i{}_j + L\,\delta^i{}_j \tag{2.15}$$

Also, any vector operator ζ^i (such as $\xi_1{}^i$ or $g^{ij}\eta_{PQj}$) can be separated into three parts $\zeta_0{}^i$, $\zeta_+{}^i{}_j$ and $\zeta_-{}^i{}_j$, of which the first commutes with L, the second increases the eigenvalue of L by one unit, and the third decreases the eigenvalue of L by one unit, thus:

$$\zeta^i = \zeta_0^i + \zeta_+^i + \zeta_-^i$$
$$\zeta_0^i = -[L(L+1)]^{-1} l_+{}^i{}_j l_-{}^j{}_k \zeta^k$$
$$\zeta_+{}^i = [L(2L+1)]^{-1} l_0{}^i{}_j l_-{}^j{}_k \zeta^k$$
$$\zeta_-{}^i = [(L+1)(2L+1)]^{-1} l_0{}^i{}_j l_+{}^j{}_k \zeta^k \tag{2.16}$$

We can effect such a separation for each of the vector operators $\xi_1{}^i{}_I$ and η_{PQu} in (2.11), and it is then easy to separate from the product

$$\Pi_{uv..} = \eta_{PQu}\eta_{RSv}\ldots \tag{2.17}$$

the part $\Pi_{\beta uv..}$ which increases the eigenvalue of L by l_β units, and from

$$\Pi^{ij..} = \xi_1{}^i{}_I \xi_1{}^j{}_J \ldots \tag{2.18}$$

the part $\Pi_\alpha{}^{ij..}$ which increases the eigenvalue of L by $l - l_\beta$ units. Then the tensor

$$U_{\alpha\beta}{}^{ij..}{}_{uv..} = \Pi_\alpha{}^{ij..}\,\Pi_{\beta uv..} S' \tag{2.19}$$

will be an eigenvector of L corresponding to the eigenvalue l, and moreover provides an irreducible representation of $SO(3)$, within the irreducible representation of $U(3)$, labelled $(l_1, l_2, l_3, l_\beta, l)$.

The structure of the tensor $U_{\alpha\beta}$ so defined is important for our subsequent considerations. It has $m = l_1 - l_2$ superscripts $i, j, k, l, \ldots$, and $n = l_2 - l_3$ subscripts $u, v, w, x, \ldots$. We can reduce the number of subscripts to l_β by multiplying by $\frac{1}{2}(n - l_\beta)$ factors $g^{uv}, g^{wx}, \ldots$; but multiplication by a further factor of this type would cause the tensor to vanish identically. Also, suppose $l_\alpha = l - l_\beta$ when $m + n - l$ is even, but $l_\alpha = l - l_\beta + 1$ when $m + n - l$ is odd.

Then we can reduce the number of superscripts by multiplying the tensor by $\frac{1}{2}(m - l_\alpha)$ factors $g_{ij}, g_{kl}, \ldots$, but multiplication by an additional factor of this kind would cause the tensor to vanish identically. Suppose $V^{ij..}{}_{uv..}$ is the tensor with l_α superscripts and l_β subscripts obtained by contraction with the metric tensor. Then

$$U_{\alpha\beta}{}^{ij..}{}_{uv..} = V^{cd..}{}_{pq..}\, G^{ij..pq}{}_{cd..uv..} \tag{2.20}$$

where G is a purely numerical tensor constructed from the metric tensor, and having the same properties as $U_{\alpha\beta}$, so far as the free subscripts $u, v, \ldots$ are concerned. For instance, if $m = n = 2$, $l_\alpha = 2$ and $l_\beta = 0$,

$$\begin{aligned} 12G^{ij}{}_{cduv} = {} & 4\delta^i{}_c\delta^j{}_d g_{uv} - (\delta^i{}_u\delta^j{}_c + \delta^j{}_u\delta^i{}_c)g_{dv} \\ & - (\delta^i{}_v\delta^j{}_c + \delta^j{}_v\delta^i{}_c)g_{du} + g^{ij}g_{cu}g_{dv} \end{aligned}$$

and if $m = n = 2$, but $l_\alpha = 0$ and $l_\beta = 2$,

$$\begin{aligned} 12G^{ijpq}{}_{uv} = {} & 4\delta^p{}_u\delta^q{}_v g^{ij} - (\delta^i{}_u\delta^p{}_v + \delta^p{}_u\delta^i{}_v)g^{jq} \\ & - (\delta^j{}_u\delta^q{}_v + \delta^q{}_u\delta^j{}_v)g^{ip} + g^{ip}g^{jq}g_{uv} \end{aligned} \tag{2.21}$$

Obviously G always has l_β dummy subscripts $p, q, \ldots$ and m free superscripts $i, j, \ldots$; it has l_α dummy subscripts $c, d, \ldots$ and n free subscripts $p, q, \ldots$. It should be noticed also that $l = l_\alpha + l_\beta$ when $m + n - l$ is even but $l = l_\alpha + l_\beta - 1$ when $m + n - l$ is odd, and that $m - l_\alpha$ and $n - l_\beta$ are always even. When $m + n - l$ is odd, it is necessary that $l \geqslant 1$, $l_\alpha \geqslant 1$ and $l_\beta \geqslant 1$.

The numbers l_α and l_β may be interpreted as angular momenta associated with the superscripts $i, j, \ldots$ and the subscripts $u, v, \ldots$ respectively; because of the condition (2.12), the composition of these angular momenta can only result in a state of angular momentum $l = l_\alpha + l_\beta$ (when $l_1 - l_3 - l$ is even) or $l = l_\alpha + l_\beta - 1$ (when $l_1 - l_3 - l$ is odd). Since l_α and l_β are not independent quantum numbers for a given value of l, we avoid giving preference to one of them by defining

$$Z = l_\alpha - l_\beta \tag{2.22}$$

and shall refer to this quantum number subsequently as the Z-spin. Together with l and the usual magnetic quantum number (component of the angular momentum), it specifies completely the state of angular momentum within a given irreducible representation of $U(3)$.

The above is not yet a solution of the problem outlined in the Introduction, because we have not defined, even implicitly, an operator whose eigenvalue is the Z-spin. However, it does provide a means of defining and labelling the irreducible representations of $SO(3)$, which is not restricted to the application so far considered. Simply stated, suppose that we have a tensor $U_c{}^{ij..}{}_{uv..}$ which is totally symmetric in $m = l_1 - l_2$ superscripts and $n = l_2 - l_3$ subscripts, and satisfies the condition (2.12), so that it belongs to the irreducible representation of $U(3)$ labelled (l_1, l_2, l_3), or the irreducible representation of $SU(3)$ labelled (m, n). Then we represent this tensor in the form

$$U_c{}^{ij..}{}_{uv..} = \Pi^{ij..}\Pi_{uv..}S \tag{2.23}$$

To obtain the irreducible representations of $SO(3)$ contained in this tensor, we simply separate from $\Pi_{uv..}$ the part $\Pi_{\beta uv}$ which makes $\Pi_{\beta uv..}S'$ an eigenvector of L corresponding to the eigenvalue l_β, and then separate from $\Pi^{ij..}\Pi_{\beta uv..}S'$ the part $U_{\alpha\beta}{}^{ij..}{}_{uv..}$ which is an eigenvector of L corresponding to the eigenvalue l. In the following section we shall examine the application of this procedure to problems of particle physics, and in particular to the quark model of the hadrons.

3. *Applications to Particle Physics*

The more successful theories of the hadrons at present are based on the classification of both baryons and mesons in multiplets and submultiplets which correspond to representations of subgroups of $U(3)$. In Gell-Mann's well-known scheme (Gell-Mann, 1962) the multiplets correspond to representations of $SU(3)$, and the submultiplets to representations of $SU(2)$, with labelling parameters I and Y which are identified with the isospin and hypercharge. This is, however, not the only possibility, and the authors (Bracken & Green, 1973) have recently pointed out the advantages of a quark model in which the isospin and hypercharge are related to the decomposition of $U(3)$ into representations of $SO(3)$. We shall show that the problem of defining I and Y in such a theory is effectively reduced to an analogue of the problem formulated in the previous section.

In this section we are concerned with second quantisation, and, instead of the coordinates $\xi_I{}^i$, introduce a set of creation operators $a_P{}^p$ and b_{Uu} ($p, u = 1, 2, 3; P = 1, 2, \ldots M; U = 1, 2, \ldots N$) and the corresponding annihilation operators a_{Pp} and $b_U{}^u$, which are hermitean conjugates of $a_P{}^p$ and b_{Uu}, respectively. They may be either boson or fermion operators, satisfying

$$[a_{Pp}, a_Q{}^q]_\pm = \delta_{PQ}\delta^q{}_p$$
$$[b_U{}^u, b_{Vv}]_\pm = \delta_{UV}\delta^u{}_v \tag{3.1}$$

where

$$[A, B]_+ = \{A, B\} = AB + BA$$
$$[A, B]_- = [A, B] = AB - BA \tag{3.2}$$

Although the fermion operators would seem to be more appropriate to the quark model we have in mind, the boson operators have the mathematical advantage that all finite dimensional representations of $U(3)$ can be constructed with the help of only one pair of operators a_p, a^p, b_u and b^u of each kind.

The generators of $U(3)$, in the type of representation now considered, are given by

$$a^i{}_j = \Sigma_P a_P{}^i a_{Pj} - \Sigma_U b_{Uj} b_U{}^i + \lambda_2 \delta^i{}_j$$
$$\lambda_2 = \Sigma_U b_{Ui} b_U{}^i \tag{3.3}$$

It is easy to verify that the scalar λ_2 commutes with $a^i{}_j$, and that these generators, like those defined in (2.2), satisfy the required commutation relations

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j \tag{3.4}$$

Also, if we define λ_1 and λ_3 by

$$\begin{aligned} \lambda_1 + \lambda_3 &= \Sigma_P a_P{}^i a_{Pi} + \lambda_2 \\ (\lambda_1 + 2)\lambda_3 &= \Sigma_P \Sigma_U a_P{}^i b_{Ui} a_{Pj} b_U{}^j \end{aligned} \tag{3.5}$$

the invariants λ_1, λ_2 and λ_3 will satisfy the same relations (2.3) as before. We may also define the operators

$$\begin{aligned} \mu &= \lambda_1 - \lambda_2 = \Sigma_P a_P{}^i a_{Pi} - \lambda_3 \\ \nu &= \lambda_2 - \lambda_3 = \Sigma_U b_{Ui} b_U{}^i - \lambda_3 \end{aligned} \tag{3.6}$$

whose eigenvalues $m = l_1 - l_2$ and $n = l_2 - l_3$ label representations of $SU(3)$.

We now consider the problem of constructing irreducible representations of $U(3)$, and first introduce a 'vacuum state' vector $\rangle$ satisfying

$$a_{Pp}\rangle = 0; \qquad b_U{}^u\rangle = 0 \tag{3.7}$$

for all values of P, p, U and u; this vector obviously belongs to the representation (0, 0, 0) of $U(3)$ in which the eigenvalues of λ_1, λ_2 and λ_3 are all zero. A scalar factor like $\sigma_{PU} = a_P{}^i b_{Ui}$ commutes with $a^i{}_j - \lambda_2 \delta^i{}_j$ but increases the eigenvalue l_2 of λ_2 by 1; hence if

$$S = \sigma_{PU}\,\sigma_{QV} \ldots \tag{3.8}$$

is a product of l_3 such factors, the vector $S\rangle$ will belong to the representation (l_3, l_3, l_3) of $U(3)$. We now have to find operators which increase the eigenvalues of λ_1 and λ_2 but leave l_3 unchanged. The operators $a_P{}^p$ and b_{Uu} will not do as they stand, because they do not commute with λ_3 as defined in (3.3). However, the matrix elements of the modified creation operators

$$\begin{aligned} A_P{}^p &= [(\lambda_1 + \lambda_3 + 1 - l_3) - (\lambda_1 + 2)\lambda_3] a_P{}^p + a_P{}^p(\lambda_1 + 2)\lambda_3 \\ B_{Uu} &= [(\lambda_1 + \lambda_3 + 1 - l_3) - (\lambda_1 + 2)\lambda_3] b_{Uu} + b_{Uu}(\lambda_1 + 2)\lambda_3 \end{aligned} \tag{3.9}$$

between two eigenvectors of λ_3 corresponding to the eigenvalues l_3 and $l_3 + 1$, respectively, obviously vanish, so that any product of these operators will not change the eigenvalue l_3 of λ_3. Also, these operators can be evaluated explicitly in terms of the a's and b's, with the help of (3.5). Thus, to form a vector belonging to the representation (l_1, l_2, l_3) of $U(3)$, we need only to apply a polynomial operator of the $(l_1 - l_2)$th degree in the A's and the $(l_2 - l_3)$th degree in the B's, to the vector $S\rangle$ belonging to the representation (l_3, l_3, l_3). This polynomial must, however, have certain symmetries in its superscripts and subscripts. Let

$$\Pi^{pq\cdots} = S_{PQ\cdots}(A_P{}^p A_Q{}^q \ldots) \tag{3.10}$$

represent the result of symmetrising, or anti-symmetrising the product $A_P{}^p A_Q{}^q \ldots$ with respect to the subscripts $P, Q, \ldots$, depending on whether the creation operators commute or anticommute; then $\Pi^{pq\cdots}$ will be symmetric in its $(l_1 - l_2)$ superscripts. Similarly,

$$\Pi_{uv\cdots} = S_{Uv\cdots}(B_{Uu} B_{Vv} \ldots) \tag{3.11}$$

will be symmetric in its $(l_2 - l_3)$ subscripts. Then the tensor

$$U_c{}^{pq\cdots}{}_{uv\cdots} = \Pi^{pq\cdots}\Pi_{uv\cdots}S\rangle \tag{3.12}$$

will belong to the representation (l_1, l_2, l_3) of $U(3)$. Because of this, the identity (2.12) is automatically satisfied, as one can verify directly with the help of (3.9) if desired.

The tensor $U_c{}^{pq\cdots}{}_{uv\cdots}$ defined by (3.12) can now be decomposed into components irreducible with respect to $SO(3)$ by the method of the previous section. The operator L is again defined by (2.13), and $l_0{}^i{}_j$, $l_+{}^i{}_j$ and $l_-{}^i{}_j$ by (2.15). With the help of these operators, we can separate the creation operators $A_P{}^i$ and B_{Ui} into parts which change the eigenvalue of L by 0, +1 and −1 respectively:

$$\begin{aligned} A_P{}^i &= A_{P0}{}^i + A_{P+}{}^i + A_{P-}{}^i \\ B_{Ui} &= B_{U0i} + B_{U+i} + B_{U-i} \end{aligned} \tag{3.13}$$

where

$$A_{P0}{}^i = -[L(L+1)]^{-1} l_+{}^i{}_j l_-{}^j{}_k A_P{}^k \tag{3.14}$$

etc. If we substitute these identities into $\Pi^{pq\cdots}$ and $\Pi_{uv\cdots}$, as defined in (3.10) and (3.11), it is a simple matter to separate from Π_{uv} the component $\Pi_{\beta uv\cdots}$ which increases the eigenvalue of L by l_β, and from $\Pi^{pq\cdots}$ the component $\Pi_\alpha{}^{pq\cdots}$ which increases the eigenvalue of L by $l - l_\beta$. Then

$$U_{\alpha\beta}{}^{pq\cdots}{}_{uv\cdots} = \Pi_\alpha{}^{pq\cdots}\Pi_{\beta uv\cdots}S\rangle \tag{3.15}$$

belongs to the irreducible representation of $SO(3)$ within $U(3)$ labelled $(l_1, l_2, l_3, l_\beta, l)$. As we have seen in the previous section, it is necessary that l_β should differ from $n = l_2 - l_3$ by an even integer; this corresponds to the well-known fact that a completely symmetric tensor like $\Pi_{uv\cdots}S\rangle$, of rank n, yields only the angular momenta $n, n-2, \ldots$. Thus, on substitution from (3.13), terms with factors like B_{U0i}, with the subscript 0, which increase the rank but do not change the angular momentum, must vanish; and terms contributing to $\Pi_{\beta uv\cdots}S\rangle$, must have l_β unpaired factors like B_{U+i}, and $n - l_\beta$ paired factors like B_{U-i} and B_{U+j}. The fact that $\Pi^{pq\cdots}$ is completely symmetric in its superscripts, and the condition (3.12), have similar consequences. On substitution from (3.13) into $\Pi^{pq\cdots}\Pi_{\beta uv\cdots}S\rangle$, terms with more than one factor like $A_{P0}{}^i$ will certainly vanish, and one such factor will be found in each non-vanishing term of $\Pi_\alpha{}^{pq\cdots}\Pi_{\beta uv\cdots}S\rangle$ if and only if $m + n - l$ is odd. Also, there will be l_α unpaired factors like $A_{P+}{}^i$, and $m - l_\alpha$ paired factors like $A_{P-}{}^i$ and

$A_{P+}{}^j$, in each non-vanishing term. Again we note that when $m + n - l$ is odd, $l_\alpha \geqslant 1$ and $l_\beta \geqslant 1$, so that $l \geqslant 1$.

We are now in a position to explain the use of the quantum numbers l_β and l (or Z and l) in labelling submultiplets of $SU(3)$, so as to correspond to the isospin multiplets in the usual classification of the hadrons. This is certainly an interesting alternative to the usual method of labelling the submultiplets with the eigenvalues of invariants derived from the decomposition of representations of $SU(3)$ into components irreducible with respect to $SU(2)$. It is feasible because the enveloping algebra of $SO(3)$ contains a subalgebra equivalent to $SU(2)$; for convenience, we summarise a proof of this in the following paragraph.

Let us choose a cartesian basis, so that the metric tensor is $g_{ij} = \delta_{ij}$; then, since $l_x = -il^2{}_3$, $l_y = -il^3{}_1$ and $l_z = -il^1{}_2$ are the generators of rotations about orthogonal axes, the operators

$$\theta_x = \exp(i\pi l_x), \qquad \theta_y = \exp(i\pi l_y)$$
$$\theta_z = \exp(i\pi l_z) \tag{3.16}$$

effect rotations of 180° about three orthogonal axes. Thus, they satisfy $\theta_x{}^2 = 1$, $\theta_y\theta_z = \theta_z\theta_y = \theta_x$, $\{\theta_x, l_y\} = \{\theta_x, l_z\} = 0$, and similar relations obtained by cyclic interchange of x, y and z. So, if

$$k_x = l_x\theta_x, \qquad k_y = il_y\theta_x, \qquad k_z = l_z$$
$$K = \tfrac{1}{2} - (k_x + k_y + k_z) \tag{3.17}$$

we have

$$\{k_x, k_y\} = k_z, \qquad \text{etc.}$$

and

$$K^2 = k_x{}^2 + k_y{}^2 + k_z{}^2 + \tfrac{1}{4} = (L + \tfrac{1}{2})^2 \tag{3.18}$$

It is clear from (3.17) that when the eigenvalue l of L is zero, the eigenvalue k of K is $\frac{1}{2}$; but, when $l \neq 0$, there are two eigenvalues $k = l + \frac{1}{2}$ and $k = -(l + \frac{1}{2})$. Also, if

$$c_x = (\tfrac{1}{2} - k_x)\theta_x, \qquad c_y = (\tfrac{1}{2} - k_y)\theta_y$$
$$c_z = (\tfrac{1}{2} - k_z)\theta_z \tag{3.19}$$

then c_x, c_y and c_z commute with K, and satisfy

$$\{c_x, c_y\} = c_z, \qquad \{c_y, c_z\} = c_x, \qquad \{c_z, c_x\} = c_y$$
$$c_x{}^2 + c_y{}^2 + c_z{}^2 = K(K + 1) \tag{3.20}$$

The representations of the algebra generated by c_x, c_y and c_z are labelled by the eigenvalues of K, and, as we have just seen, there are two of these for $l \neq 0$, and one for $l = 0$. As l_z takes integral eigenvalues between $-l$ and $+l$, c_z takes eigenvalues in the sequence $\frac{1}{2}, -\frac{3}{2}, +\frac{5}{2}, -\frac{7}{2}, \ldots$, terminating in

$(-1)^l(l+\frac{1}{2})$ when $k = l + \frac{1}{2}$, and $(-1)^{l-1}(l-\frac{1}{2})$ when $k = -(l+\frac{1}{2})$. Any representation of c_x, c_y and c_z is therefore $(2I+1)$-dimensional, where

$$I = \tfrac{1}{2}|k + \tfrac{1}{2}| - \tfrac{1}{2} \tag{3.21}$$

Clearly, $I = \frac{1}{2}l$ or $\frac{1}{2}(l-1)$ according as $k > 0$ or $k < 0$. We have considered the problem of defining operations I_1, I_2, I_3, representing the isospin components, elsewhere (Bracken & Green, 1973), and have pointed out that for most purposes the operators c_x, c_y and c_z satisfying (3.20) will serve as well.

We shall adopt (3.21) as our definition of the isospin, and remark again that each irreducible representation of $SO(3)$ with $l \neq 0$ contains two representations of the operators c_x, c_y and c_z, corresponding to the values $I = \frac{1}{2}l$ and $I = \frac{1}{2}(l-1)$; but when $l = 0$, only the value $I = 0$ is allowed. To verify that this definition of the isospin is consistent with that normally used in classifying submultiplets of $SU(3)$, we need also to find an expression for the hypercharge Y. This takes values differing by an integer, with a minimum value of $-(2m+n)/3$ and a maximum of $(m+2n)/3$, in the representation of $SU(3)$ labelled (m, n). For a given value of Y, I takes values differing by an integer, between a minimum value of $\frac{1}{2}|Y + 2(m-n)/3|$ and a maximum value of $\frac{1}{2}(m+n) - \frac{1}{2}|Y - (m-n)/3|$. Since l_α takes the values $m, m-2, \ldots$, and l_β takes the values $n, n-2, \ldots$, these requirements are met by taking

$$\begin{aligned} Y + 2(m-n)/3 &= l_\alpha - l_\beta + \Delta \\ \Delta &= \pm(2l - l_\alpha - l_\beta - 2I) \end{aligned} \tag{3.22}$$

where the positive sign is to be chosen if $l_\alpha < l_\beta$, and the negative sign if $l_\alpha \geqslant l_\beta$. It will be seen that since $2I$ is equal to l or $l-1$, and $l_\alpha + l_\beta$ is equal to l or $l+1$, Δ has values 0 or ± 1, with exceptions when $l_\alpha = 0$ or $l_\beta = 0$, since then $l_\alpha + l_\beta$ cannot have the value $l+1$.

The above formula for Y may seem rather complicated, but this simply reflects the fact that l_α, l_β and the sign of k are more appropriate labels for the isospin submultiplets when the $SU(3)$ multiplets are decomposed with respect to $SO(3)$. Among the well-known baryons ($m = l_\alpha = 1$; $n = l_\beta = 1$), the Λ-particle joins the nucleons in an $SO(3)$ multiplet with $l = 1$, while the other members of the octet form an $SO(3)$ multiplet with $l = 2$; the sign of k determines the strangeness. There may be some advantages in a quark model based on this scheme. As in Gell-Mann's scheme, the quarks are distinguished by the values 1, 2, 3 taken by the tensor subscripts and superscripts; but, as is evident from (3.17), the quark corresponding to the value 3 necessarily plays a somewhat different role from the others. As we have pointed out elsewhere (1973), the decomposition with respect to $SO(3)$ is required if the quarks are to satisfy a simple kind of parastatistics.

4. *The Labelling Operators*

Our object in this section is to define, at least implicitly, operators L_α and L_β whose eigenvalues, in an irreducible representation of $SO(3)$ within $U(3)$, coincide with the integers l_α and l_β introduced in Sections 2 and 3 to label such representations. It is sufficient to consider irreducible representations of $U(3)$ labelled $(m, 0, -n)$, where $m = l_1 - l_2$ are $n = l_2 - l_3$ in the notation adopted previously. [Hughes (1973a, b) writes $p = m + n$ and $q = n$.] All $SO(3)$ invariants in this representation are scalars constructed from the tensors $b^i{}_j$ are $\bar{b}^i{}_j$ defined by

$$b^i{}_j = a^i{}_j - \lambda_2 \delta^i{}_j$$
$$\bar{b}^i{}_j = g^{il} b^k{}_l g_{kj} \tag{4.1}$$

and are therefore expressible as traces of matrix products with factors b and $\bar{b}$, the matrices whose elements are b^i_j and $\bar{b}^i_j$ respectively. We shall denote these traces by

$$\langle b \rangle = b^i{}_i = \langle \bar{b} \rangle, \qquad \langle b^2 \rangle = b^i{}_j b^j{}_i$$
$$\langle b\bar{b} \rangle = \langle \bar{b}b \rangle = \bar{b}^i{}_j b^j{}_i, \text{ etc.}$$

The operators μ and ν whose eigenvalues are m and n may be defined by

$$\langle b \rangle = \mu - \nu$$
$$\langle b^2 \rangle = \mu(\mu + 2) + \nu(\nu + 2) \tag{4.2}$$

according to (2.3) and (3.6), and the 'total angular momentum' L is given by

$$\langle b^2 - b\bar{b} \rangle = L(L + 1) \tag{4.3}$$

according to (2.13). The operators x and y defined by Racah (1964) are

$$x = \langle b\bar{b}b + \bar{b}b\bar{b} \rangle + \langle b \rangle(4L(L + 1)/3 - 3\langle b^2 \rangle + \langle b \rangle^2 - 2)$$
$$y = 8\langle b \rangle x/3 - 2\langle b\bar{b}^2 b + \bar{b}b^2\bar{b} \rangle - (16\langle b^2 \rangle/9 + 15)L(L + 1)$$
$$+ 2\langle b^2 \rangle(\langle b^2 \rangle + 2\langle b \rangle^2 + 5) + 2\langle b \rangle^2(1 - \langle b \rangle^2)$$

in this notation, and the operators $O_l{}^0$ and $Q_l{}^0$ introduced by Hughes (1973a, b) are†

$$O_l{}^0 = -3\sqrt{(6)}x$$
$$Q_l{}^0 = -18y - 12L(L + 1)(L + L + 3)$$
$$+ 24L(L + 1)(\langle b^2 \rangle - \langle b \rangle^2/3)$$

What we would like is a formula for L_α or L_β in terms of

$$S_3 = \langle b\bar{b}b + \bar{b}b\bar{b} \rangle$$
$$S_4 = \langle b\bar{b}^2 b + \bar{b}b^2\bar{b} \rangle \tag{4.4}$$

or, equivalently, in terms of x and y, or $O_l{}^0$ and $Q_l{}^0$. It should be pointed out that L_α and L_β do not commute with S_3 and S_4, which latter pair also do not commute with one another, so that there is no possibility of finding a relation

† The second of these results is the transcription of a formula kindly supplied to the authors by Dr. Hughes.

between L_α or L_β and S_3 alone; as we shall see, it is not even true that S_3 and S_4 are hermitean in a representation in which L_α and L_β are diagonal.

Applied to a tensor $U^{pq\cdot\cdot}{}_{uv\cdot\cdot}$ of contravariant rank m and covariant rank n, the generator $b^i{}_j$ yields

$$\begin{aligned} b^i{}_j U^{pq\cdot\cdot}{}_{uv\cdot\cdot} &= (\alpha^i{}_j - \beta^i{}_j) U^{pq\cdot\cdot}{}_{uv\cdot\cdot} \\ \alpha^i{}_j U^{pq\cdot\cdot}{}_{uv\cdot\cdot} &= \delta^p{}_j U^{iq\cdot\cdot}{}_{uv\cdot\cdot} + \delta^q{}_j U^{pi\cdot\cdot}{}_{uv\cdot\cdot} + \ldots \\ \beta^i{}_j U^{pq\cdot\cdot}{}_{uv\cdot\cdot} &= \delta^i{}_u U^{pq\cdot\cdot}{}_{jv\cdot\cdot} + \delta^i{}_v U^{pq\cdot\cdot}{}_{uj\cdot\cdot} + \ldots \end{aligned} \tag{4.5}$$

where $\alpha^i{}_j$ and $-\beta^i{}_j$ are operators defined initially by tensor correspondences, but can be regarded as commuting generators of unitary groups, since they satisfy the same commutation relations as $b^i{}_j$. They are obviously related to the creation and annihilation operators $a_P{}^p$, b_{Uu} and a_{Pp}, $b_U{}^u$ introduced in the previous section, and have a representation

$$\begin{aligned} \alpha^i{}_j &= a^i a_j = a_j a^i - \delta^i{}_j \\ \beta^i{}_j &= b_j b^i = b^i b_j - \delta^i_j \end{aligned} \tag{4.6}$$

in terms of the commuting boson creation and annihilation operators a^p, b_u and a_p, b^u. The ten scalars $\frac{1}{2}\{a^i, a_i\}$, $\frac{1}{2}\{b^i, b_i\}$, $a^i b_i$, $b^i a_i$, $a^i a^i$, $b^i b^i$, $a^i b^i$, $a_i a_i$, $b_i b_i$ and $a_i b_i$ are generators of $Sp(4)$ in a representation equivalent to the representation of $SO(2, 3)$ labelled $(\frac{1}{2}l + \frac{3}{2}, \frac{1}{2}l)$, where l is again the eigenvalue of L. These operators are defined not on a single irreducible representation of $U(3)$, but on a set of such representations labelled $(m + n + l_3, n + l_3, l_3)$, where l_3 takes arbitrary non-negative integral values. This corresponds to the fact that the form of the tensor $U^{pq\cdot\cdot}{}_{uv\cdot\cdot}$ provides no explicit indication of the value of l_3. In the reducible representation in which l_3 takes all non-negative integral values,

$$\begin{aligned} \langle\alpha\rangle &= \alpha^i{}_i = \mu + \lambda_3 \\ \langle\beta\rangle &= \beta^i{}_i = \nu + \lambda_3 \\ \langle\alpha\beta\rangle &= \alpha^i{}_j\beta^j{}_i = \lambda_3(\mu + \nu + \lambda_3 + 2) \end{aligned} \tag{4.7}$$

Let us introduce operators Λ_α, Λ_β defined by

$$\begin{aligned} \Lambda_\alpha(\Lambda_\alpha + 1) &= \langle\alpha^2 - \alpha\bar{\alpha}\rangle = \alpha^i{}_j(\alpha^j{}_i - \bar{\alpha}^j{}_i) \\ \Lambda_\beta(\Lambda_\beta + 1) &= \langle\beta^2 - \beta\bar{\beta}\rangle \end{aligned} \tag{4.8}$$

where $\bar{\alpha}^j{}_i = g^{jk} g_{il} \alpha^l{}_k$, etc. Clearly Λ_α and Λ_β are angular momenta associated with contravariant and covariant tensors respectively, and take non-negative integral eigenvalues which we may identify with the labelling parameters l_α and l_β. But, as can be seen from the fact that Λ_α and Λ_β do not commute with $\langle\alpha\beta\rangle$, they are not defined on irreducible representations of $U(3)$, and cannot therefore be identified with the labelling operators L_α and L_β. To establish the relation between Λ_α, Λ_β and L_α, L_β, let us consider a representation in which the commuting operators $\langle\alpha\rangle$, $\langle\beta\rangle$, Λ_α and Λ_β are diagonal, and introduce a similarity transformation $\langle\alpha\beta\rangle \to S\langle\alpha\beta\rangle S^{-1}$, such that $S\langle\alpha\beta\rangle S^{-1}$ is also diagonal, and therefore commutes with Λ_α and Λ_β. Since $\langle\alpha\beta\rangle$ commutes with $\langle\alpha\rangle$ and $\langle\beta\rangle$, S may be chosen so that $\langle\alpha\rangle$ and $\langle\beta\rangle$ remain diagonal under this transformation. Then the operators

$$L_\alpha = S^{-1}\Lambda_\alpha S, \qquad L_\beta = S^{-1}\Lambda_\beta S \tag{4.9}$$

will commute with $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\alpha\beta\rangle$; and, as they have the same eigenvalues l_α and l_β as Λ_α and Λ_β, they are the required labelling operators. The explicit determination of S remains an unsolved problem. Since there are no more than five independent commuting operators in the relevant representation of $SO(2, 3)$, there is a general relation between $L_\alpha, L_\beta, L, \langle\alpha\rangle, \langle\beta\rangle$ and $\langle\alpha\beta\rangle$, which as we know from our previous considerations, implies that $l_\alpha + l_\beta = l$ or $l + 1$, according as $m + n - l$ is even or odd.

In the following, we intend to exploit the fact that the tensor representations of $U(3)$ allow arbitrary non-negative integral values of l_3, and evaluate S_3 and S_4, as defined in (4.4), by adopting provisionally a representation in which Λ_α, Λ_β, $\mu + \lambda_3$ and $\nu + \lambda_3$ are diagonal. Then we may implicitly apply the similarity transformation (S) which leaves L_α, L_β and λ_3 diagonal, and finally project on to the subspace corresponding to $l_3 = 0$. In this way, we obtain expressions for S_3 and S_4 involving L_α and L_β, and two operators X, $\bar{X}$ which change the eigenvalues of both L_α and L_β by two units:

$$\begin{aligned} XL_\alpha &= (L_\alpha - 2)X; \qquad & XL_\beta &= (L_\beta + 2)X \\ \bar{X}L_\alpha &= (L_\alpha + 2)\bar{X}; \qquad & \bar{X}L_\beta &= (L_\beta - 2)\bar{X} \end{aligned} \tag{4.10}$$

Finally, it is possible to eliminate X and $\bar{X}$ from these expressions for S_3 and S_4, and thus obtain a relation between L_α, S_3 and S_4 which implicitly defines the operator L_α in terms of the generators of $U(3)$.

In evaluating S_3 and S_4, we make use of the definitions (4.5), which imply that

$$\begin{aligned} \alpha^i{}_j &= \delta^p{}_j \begin{bmatrix} p \\ i \end{bmatrix} \\ \beta^i{}_j &= \delta^i{}_u \begin{bmatrix} u \\ j \end{bmatrix} \end{aligned} \tag{4.11}$$

where $\begin{bmatrix} p \\ i \end{bmatrix}$ represents the effect of replacing the superscript p of the tensor $U^{pq\cdot\cdot}{}_{uv\cdot\cdot}$ by i, and there is an implicit summation over all superscripts; similarly, $\begin{bmatrix} u \\ j \end{bmatrix}$ represents the effect of replacing the subscript u by j, and there is an implied summation over all subscripts. We shall also write

$$\begin{pmatrix} pq\,.\,.\; & uv\,.\,. \\ ij\,.\,.\; & kl\,.\,. \end{pmatrix} = PS^{-1}\left(\begin{bmatrix} p \\ i \end{bmatrix}\begin{bmatrix} q \\ j \end{bmatrix}\cdots\begin{bmatrix} u \\ k \end{bmatrix}\begin{bmatrix} v \\ l \end{bmatrix}\cdots\right)S \tag{4.12}$$

to represent the effect of applying the similarity transformation to the result of any sequence of such substitutions, and then the projection P on to the irreducible representation $l_3 = 0$. With this notation, the eigenvalues of $\begin{pmatrix} p \\ p \end{pmatrix}, \begin{pmatrix} u \\ u \end{pmatrix}, \begin{pmatrix} pq \\ pq \end{pmatrix}, \ldots$ are obviously $m, n, m(m-1), \ldots$; hence

$$\begin{aligned}
\langle \alpha \rangle' &= PS^{-1}\langle \alpha \rangle S = \mu, \qquad \langle \beta \rangle' = \nu \\
\langle \alpha^2 \rangle' &= \begin{pmatrix} pq \\ pq \end{pmatrix} + 3\begin{pmatrix} p \\ p \end{pmatrix} = \mu(\mu + 2) \\
\langle \alpha\beta \rangle' &= \delta^p{}_u \begin{pmatrix} pu \\ ii \end{pmatrix} = 0
\end{aligned} \tag{4.13}$$

where the prime denotes the effect of the similarity transformation and projection. We may define L_α and L_β, according to (4.8) and (4.9), by means of

$$M = \langle \alpha(\bar{\alpha} - 1) \rangle' = g^{pq}\begin{pmatrix} pq \\ ii \end{pmatrix} = \mu(\mu + 1) - L_\alpha(L_\alpha + 1)$$

$$N = \langle \bar{\beta}(\beta - 1) \rangle' = g_{uv}\begin{pmatrix} uv \\ ii \end{pmatrix} = \nu(\nu + 1) - L_\beta(L_\beta + 1) \tag{4.14}$$

and it follows that, as expected, l_α and l_β may be interpreted as the number of unpaired superscripts and subscripts of the tensor $U^{pq\cdot\cdot}{}_{uv\ldots}$. By similar combinatorial reasoning, or using (4.3) and (4.5), we have also

$$Q = \langle \alpha\bar{\beta} \rangle' = \begin{pmatrix} up \\ pu \end{pmatrix} = \tfrac{1}{2}[L(L + 1) - L_\alpha(L_\alpha + 1) - L_\beta(L_\beta + 1)] \tag{4.15}$$

Further,

$$M_1 = \langle (\bar{\alpha} - 1)\alpha\beta \rangle' = \delta^p{}_u \begin{pmatrix} pqu \\ iiq \end{pmatrix} = (\mu - L_\alpha)(\nu + Q) + \bar{X}$$

$$N_1 = \langle \alpha(\beta - 1)\bar{\beta} \rangle' = \delta^p{}_u \begin{pmatrix} puv \\ vii \end{pmatrix} = (\nu - L_\beta)(\mu + Q) + X$$

$$\begin{aligned}
Q_1 &= \langle (\bar{\alpha} - 1)\alpha(\beta - 1)\bar{\beta} \rangle' = \delta^p{}_u \delta^q{}_v \begin{pmatrix} pquv \\ iijj \end{pmatrix} \\
&= (\mu - L_\alpha)(\nu - L_\beta)(2Q - 3) + (\mu - L_\alpha)(N + X) \\
&\quad + (\nu - L_\beta)(M + \bar{X})
\end{aligned} \tag{4.16}$$

where the normalisation of X and $\overline{X}$ is defined by

$$X = X'(\mu - L_\alpha)(L - L_\alpha)(L - L_\alpha - 1)$$
$$X = \overline{X}'(\nu - L_\beta)(L - L_\beta)(L - L_\beta - 1)$$
$$X'\overline{X}' = 1' \tag{4.17}$$

and $1'$ is an idempotent, differing from the unit matrix only by some vanishing element corresponding to extreme values of l_α and l_β.

The above are the only independent operators arising from the evaluation of S_3 and S_4. It is a straightforward though rather tedious matter to verify that

$$\begin{aligned}\langle b\bar{b}b\rangle &= 3\mu^2 - (\mu+1)\nu + (\nu - \mu + 5)Q \\ &\quad + (\mu - 2)M - (\nu + 3)N + N_1 - M_1 \\ \langle b\bar{b}^2 b\rangle &= 2\mu^2(2\mu - 1) + 2\mu\nu(\nu + 1) + \nu - \mu \\ &\quad + 2(\mu\nu + 2\mu - 2\nu - 5)Q + (\mu^2 - 3\mu + 3)M \\ &\quad + (\nu^2 + 5\nu + 7)N - 2(\mu - 2)M_1 \\ &\quad - 2(\nu + 2)N_1 + Q_1\end{aligned} \tag{4.18}$$

and hence that

$$\begin{aligned}S_3 &= (\mu - \nu)(3\mu + 3\nu + 1 - 2Q) + (2\mu + 1)M \\ &\quad - (2\nu + 1)N + 2(N_1 - M_1) \\ S_4 &= 2\mu^2(2\mu - 1) + 2\nu^2(2\nu - 1) + 2\mu\nu(\mu + \nu + 2) \\ &\quad + 4(\mu\nu - 5)Q + 2(\mu^2 + \mu + 5)M \\ &\quad + 2(\nu^2 + \nu + 5)N - 4\mu M_1 - 2\nu N_1 + 2Q_1\end{aligned} \tag{4.19}$$

By using these results, and the relations given in (4.14)-(4.17), it is easy to verify the numerical eigenvalues given for O_l^0 and Q_l^0 by Hughes (loc. cit.).

The above results may also be written

$$\begin{aligned}S_3 &= S_3^0 + \overline{X} - X \\ S_4 &= S_4^0 - 2(2\nu + L_\alpha - \mu)X - 2(2\mu + L_\beta - \nu)\overline{X}\end{aligned} \tag{4.20}$$

where S_3^0 and S_4^0 are the polynomials in μ, ν, L, L_α and L_β derived by substituting from (4.14), (4.15) and (4.16) into (4.19). By solving these equations for X and $\overline{X}$ and using (4.17), we obtain finally

$$\begin{aligned}4(\mu + \nu + L_\alpha + L_\beta)^2 X\overline{X} &= [S_4^0 - S_4 + 2(2\mu + L_\beta - \nu)(S_3^0 - S_3)] \\ &\quad . [S_4^0 - S_4 - 2(2\nu + L_\alpha - \mu)(S_3^0 - S_3)] \\ &= 4(\mu + \nu + L_\alpha + L_\beta)^2(\mu - L_\alpha + 2) \\ &\quad \times (L - L_\alpha + 2)(L - L_\alpha + 1) \\ &\quad . (\nu - L_\beta)(L - L_\beta)(L - L_\beta - 1)\end{aligned} \tag{4.21}$$

This is the equation which implicitly defines L_α, L_β and hence Z in terms of S_3 and S_4.

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